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On an Extremal Problem of Subbotin Concerning Finite Differences and Derivatives

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1. INTRODUCTION

Recently Sharma and Tzimbalario announced in [4] a generalization of a result of Subbotin which is contained in [5]. Specifically, they consider the following problem.

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be real constants and $L_n(D)$ the constant coefficient differential operator given by

$$L_n(D) = \prod_{i=1}^{n} (D - \gamma_i),\ D = \frac{d}{dx}.$$

For a given bi-infinite sequence $\{y_k\}_{-\infty}^{\infty}$ set

$$\Delta_L{}^n y_m = \prod_{i=1}^{n} (E - e^{\gamma_i})\, y_m = \sum_{k=0}^{n} r_k E^k y_m$$

where $Ey_m = y_{m+1}$. Then for symmetric differential operators,

$$L_n(-D) = (-1)^n L_n(D),$$

they determine the exact value of the supremum of

$$\inf_{f(k)=y_k} \sup_{x\in R} | L_n(D) f(x)|, \qquad k \in \{0, \pm 1, \pm 2, \ldots,\} = Z,$$

over all sequences y_k with $|\Delta_L{}^n y_m| \leqslant 1$, $m \in Z$. The case $\gamma_i = 0$ was considered by Subbotin [5].

In this paper we provide a further generalization of Subbotin's theorem which encompasses the above result as well. This theorem relies on a recent result contained in [2] and in this sense, our subsequent remarks are an addendum to this paper.

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2. The Problem

Let $\phi(x)$ be a Pólya frequency function on $(-\infty, \infty)$. Thus we are assuming that the translation kernel, $K(x, y) = \phi(x-y)$ is totally positive on $(-\infty, \infty)$, that is,

$$K\begin{pmatrix} x_1, \ldots, x_m \\ y_1, \ldots, y_m \end{pmatrix} = \det_{i,j=1,\ldots,m} \| K(x_i, y_j) \| \geqslant 0,$$

for all $x_1 < \cdots < x_m$, $y_1 < \cdots < y_m$ and $m \geqslant 1$.

A basic property of Pólya frequency functions tells us that the support of ϕ is either a finite, half-infinite or infinite interval and $\phi(x)$ will decay exponentially fast as $x \to \pm\infty$, [1, *p.* 332].

We will call a Pólya frequency function ϕ nondegenerate if for every $N \geqslant 0$, the translates $\{\phi(x-j): |j| \leqslant N\}$ are linearly independent over $[-N-1, N+1]$.

THEOREM. *Let ϕ be a nondegenerate Pólya frequency function. Then*

$$\inf_{h \in H(d)} \sup_{x \in R} |h(x)| \leqslant \sup_{i \in Z} |d_i| \frac{1}{\int_0^1 |\psi(x)|\, dx}$$

where

$$H(d) = \left\{ h \in L^\infty(-\infty, \infty): \int_{-\infty}^{\infty} \phi(x-k)\, h(x)\, dx = d_k, k \in Z \right\}$$

and

$$\psi(x) = \sum_{-\infty}^{\infty} \phi(x-k)(-1)^k,$$

and this inequality is sharp.

Before we prove this result we will discuss the special case referred to earlier.

Let $\phi_n(x)$ be the unique solution of $L_n(D)y = 0$ satisfying the condition

$$\phi_n^{(\nu)}(0) = \delta_{\nu, n-1}, \nu = 0, 1, \ldots, n-1.$$

Then, by the definition of $\Delta_L{}^n$ it follows that

$$\Delta_L{}^n e^{\gamma_j x} = 0, j = 1, \ldots, n,$$

and consequently

$$\Delta_L{}^n f(x) = \int_{-\infty}^{\infty} B_n(x-t)\, L_n(D) f(t)\, dt \tag{1}$$

where $B_n(x) = \Delta_L{}^n \phi_n(x_+) = \sum_{k=1}^{n} r_k \phi_n(x + k)_+$, $x_+ = x$, for $x \geqslant 0$ and zero otherwise.

$B_n(t)$ is the B-spline for the differential operator L_n and it is known that $B_n(t)$ is a nondegenerate Pólya frequency function [1, Chapter 10, Sect. 4].

Now, if $f(k) = y_k$ then according to (1)

$$d_k = \Delta_L{}^n y_k = \int_{-\infty}^{\infty} B_n(k - t)\, L_n(D) f(t)\, dt.$$

Therefore the above theorem implies

$$\inf_{f(k)=y_k} \sup_{x \in R} | L_n(D) f(x)| \leqslant \sup_k | \Delta_L{}^n y_k | \frac{1}{\int_0^1 | \psi_n(x)|\, dx} \tag{2}$$

where

$$\psi_n(x) = \sum_{-\infty}^{\infty} (-1)^k B_n(x - k).$$

The function $\sum_{-\infty}^{\infty} t^k B_n(x - k)$ was studied in [2] where it was shown that for each t it has exactly one simple zero in [0, 1). When

$$L_n(D) = (-1)^n L_n(-D) \tag{3}$$

it easily follows that

$$\psi_n(1 - x) = (-1)^{n+1} \psi_n(x).$$

Thus we see that when (3) holds the only zero of $\psi_n(x)$ in [0, 1) is a simple zero at $\frac{1}{2}$ when n is even and a simple zero at 0 when n is odd. This fact allows for the simplification of the best constant in (2), see [4] for further details.

We now present a proof of the theorem.

Proof. For every $N \geqslant 0$ our hypothesis guarantees that the functions $\phi(x + N),\ldots, \phi(x - N)$ form a weak Chebyshev system on $[-N-1, N + 1]$. Consequently, according to Theorem 5 of [2] we have

$$\begin{aligned} A_N(d) &= \inf_{h \in H_N(d)} \sup_{|x| \leqslant N+1} | h(x)| \\ &\leqslant \left(\inf_{h \in H_N(e)} \sup_{|x| \leqslant N+1} | h(x)| \right) \max_{|j| \leqslant N} | d_j |, \end{aligned}$$

where

$$H_N(d) = \left\{ h \in L^\infty[-N - 1, N + 1]: \int_{-N-1}^{N+1} \phi(x - j)\, h(x)\, dx = d_j, | j | \leqslant N \right\},$$

and $e = ((-1)^{-N-1}, \ldots, (-1)^{N+1})$. We will now show that for every $d = \{d_k\}_{-\infty}^{\infty}$, $\sup_k |d_k| < \infty$,

$$\lim_{N\to\infty} A_N(d) = A(d) = \inf_{h\in H(d)} \sup_{x\in R} |h(x)| .$$

To this end, let $h_0(x) = \lambda(-1)^j, j \leqslant x < j+1$ where

$$\lambda^{-1} = \int_0^1 |\psi(x)|\, dx.$$

Then an easy computation shows that

$$\int_{-\infty}^{\infty} \phi(x-k)\, h_0(x)\, dx = (-1)^k, k \in Z.$$

Consequently, $A_N(e) < \lambda$ for all N. A standard argument using w^*-compactness in $L^1[-N-1, N+1]$ establishes the existence of a $h_N \in L^\infty[-N-1, N+1]$ with $h_N \in H_N(d)$ and

$$A_N(d) = \sup_{|x|\leqslant N+1} |h_N(x)| .$$

Moreover, since $A_N(d) \leqslant \lambda$ the sequence $h_N\chi_N$, where $\chi_N(x)$ is the characteristic function of $[-N-1, N+1]$ is a uniformly bounded sequence in $L^\infty(-\infty, \infty)$. Thus we are assured of the existence of a subsequence $\{h_{N'}(x)\}$ which converges w^* to a function $\bar{h}(x) \in L^\infty(-\infty, \infty)$, i.e.

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} f(x)\, h_N, (x)\, dx = \int_{-\infty}^{\infty} f(x)\, \bar{h}(x)\, dx$$

for all $f \in L^1(-\infty, \infty)$. Clearly $\bar{h} \in H(d)$ and $A(d) = \sup_{x\in R} |\bar{h}(x)| = \lim_{N\to\infty} A_N(d)$ as well.

Thus we have established that $A(d) \leqslant A(e)\, \| d \|_\infty$. We will be finished when we demonstrate that $A(e) = \lambda$. For this purpose, we let h be any function in $H(e)$. Then $-h(x+1)$ is also in $H(e)$. Moreover, since $H(e)$ is a convex set, the function

$$G_N(x) = \frac{1}{N} \sum_{j=1}^{N} (-1)^j\, h(x+j)$$

is again in $H(e)$ for all N. As before, since $\sup_{x\in R} |G_N(x)| \leqslant \sup_{x\in R} |h(x)|$, $\{G_N\}$ has a convergent sequence $\{G_{N'}\}$ which converges w^* to a $G \in L^\infty$. Since for every $a < b$

$$\int_a^b G_N(x+1)\, dx = -\left(1+\frac{1}{N}\right) \int_a^b G_{N+1}(x)\, dx + \frac{1}{N} \int_a^b h(x+1)\, dx$$

we have $G(x+1) = -G(x)$ and hence

$$\sup_{0 \leqslant x \leqslant 1} | G(x)| \leqslant \sup_{x \in R} | h(x)| .$$

Now,

$$\int_0^1 \psi(x)\, G(x)\, dx = \sum_{-\infty}^{\infty} (-1)^k \int_0^1 \phi(x+k)\, G(x)\, dx$$

$$= \int_{-\infty}^{\infty} \phi(x)\, G(x)\, dx = 1,$$

the last inequality follows from the fact that $G \in H(e)$. Consequently,

$$1 \leqslant \lambda^{-1} \sup_{0 \leqslant x \leqslant 1} | G(x)| \leqslant \lambda^{-1} \sup_{x \in R} | h(x)|$$

and this inequality completes the proof.

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